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Project In Mathematics

MATHM901

A Lower Bound For Biases Amongst Products of Two Primes

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Preface

This text is the result of work done, by myself, in the 2015/16 academic year. It forms the piece for submission on the MathM901 Project In Mathematics course at University College London.

As for prerequisite knowledge, it is assumed that the reader has a strong understanding of undergraduate mathematics and in particular has taken the equivalent of a course in elementary number theory.

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Chapter 1

Introduction

In this text I will be focusing on a recent paper entitled *Big Biases Amongst Products Of Two Primes* [3] by David Dummit, Andrew Granville and Kershy Kisilevsky. I will begin by presenting a proof of their result and move on to discuss a computational approach for finding particular cases in which the bias is notably large. Finally I will give a proof of a Conjecture that was stated at the end of their paper. This Theorem gives a lower bound for the largest bias that can be found.
I now wish to give a brief overview of the history in this area as I consider it to be an essential background against which to present this new result.

1.1 Historical Context

The study of prime numbers is undoubtedly one of the most romantic topics in mathematics and has long been the subject of much excitement and even philosophy. One aspect of the study of prime numbers can be thought of using the analogy of a race between two or more teams. This analogy, as we shall see, is certainly a useful one and is used to great effect in *Prime Number Races* by A. Granville and G. Martin [6].

Let us consider several teams and all primes up to a given number x. We consider each prime number in turn and, depending on its properties, allocate it to one of the teams. Consider the following situation in which we have just two teams; Team 1 and Team 3. A prime p (with the exception of 2, which we discard) falls in to just one of these teams. Either $p \equiv 1 \mod 4$, in which case it is in Team 1, or $p \equiv 3 \mod 4$ and is in Team 3. Table 1.1 shows how the primes, up to various values of x, fall into these two teams. The reader will note that Team 3 appears to be always in the lead. This was first observed by Tchébychev in 1853 when he wrote that

'There is a notable difference in the splitting of the prime numbers between the two forms 4n + 3 and 4n + 1: the first form contains a lot more than the second.'

x	Number of primes $4n+3$ up to x	Number of primes $4n+1$ up to x
100	13	11
200	24	21
300	32	29
400	40	37
500	50	44
600	57	51
700	65	59
800	71	67
900	79	74
1000	87	80
2000	155	147
5000	339	329
10000	619	609
20000	1136	1125
50000	2583	2549
100000	4808	4783
200000	9006	8977
500000	20806	20731
1000000	39322	39175
2000000	74516	74416
3000000	108532	108283
4000000	141643	141502

Table 1.1: The distribution of primes into the $4n+1$ and $4n+3$ arithmetic progressions. Calculated using SageMath (see Appendix A for code).

The situation here, however, is not as straightforward as the data would suggest. Indeed, at a mere $26,861$, Team 1 takes the lead for the first time. This is somewhat comforting in light of many well known results such as Dirichlet's infinity of primes in arithmetic progressions. Furthermore this result, also known as *The Prime Number Theorem For Arithmetic Pregressions* suggests that primes are evenly distributed among arithmetic progressions. More precisely, for appropriate values of a and b

$$\frac{\#\{ \text{ primes } qn+a \leq x\}}{\#\{ \text{ primes } qn+b \leq x\}} \to 1, \ (\text{ as } x \to \infty).$$

This does not however tell us anything about the subtleties of such a prime race. In particular it doesn't tell us whether Tchébychev's observation is true for most x or indeed just appears that way for the data that can be computed. A further result that demonstrates the complexity of this question is the following, given by Littelwood in 1914 [12].

Theorem 1.1.1 (J.E Littlewood, 1914)**.** *There are arbitrarily large value of x for which there are more primes of the form $4n+1$ up to x than of the form $4n+3$. In fact, there are arbitrarily large values of x for which*

$$\#\{\text{primes } 4n+1 \leq x\} - \#\{\text{primes } 4n+3 \leq x\} \geq \frac{1}{2}\frac{\sqrt{x}}{\log x}\log\log\log x.$$

The next consideration is whether the density of these values of x for which Team 1 is in the lead is low enough for us to say that their number becomes negligible as x goes to infinity. In 1962, Knapowski and Turán conjectured just this.

Conjecture 1.1.1. *As $X \to \infty$ the percentage of integers $x \leq X$ for which there are more primes of the form $4n+3$ up to x than of the form $4n+1$ goes to 100%.*

This Conjecture did not however stand the test of time. The great work of Riemann can be used to show that, assuming the *Generalised Riemann Hypothesis* (GRH), one has the following variant of Riemann's revolutionary formula.

$$\frac{\#\{\text{primes } 4n+3 \leq x\} - \#\{\text{primes } 4n+1 \leq x\}}{\sqrt{x}/\log x} \approx 1 + 2\sum_{\substack{\text{Real numbers } \gamma>0 \\ \text{such that } \frac{1}{2}+i\gamma \\ \text{is a zero of } L(s)}} \frac{\sin(\gamma\log x)}{\gamma}. \quad (1.1)$$

By studying this formula it was shown independently by both Kaczorowski [10] and Sarnak [14] that the Knapowski-Turán Conjecture was false or at least contradicted the GRH. More worrying was that the quantity

$$\frac{1}{X}\#\{x \leq X : \text{ there are more primes of the form } 4n+3 \text{ up to } x \text{ than of the form } 4n+1\},$$

does not tend to a limit but fluctuates as $X \to \infty$. Fortunately, the efforts of those trying to understand this trend was not in vain. All that was needed, as shown in 1994 by Rubinstein and Sarnak [13], was to consider a different way of counting the primes. The key to understanding this prime race is to weight each prime p with a value of $1/p$ instead of 1. This so called 'logarithmic measure', it turns out, provides a far more interesting perspective on prime races. This technique was used to brilliant effect in producing the following result.

Theorem 1.1.2 (Rubinstein and Sarnak, 1994)**.** *As $X \to \infty$*

$$\frac{1}{\log X}\sum_{\substack{x \leq X \\ \textit{There are more primes } 4n+3 \\ \textit{than } 4n+1 \textit{ up to } x}} \frac{1}{x} \longrightarrow 0.9959...$$

The factor in front here is necessary as a scaling factor since $\sum_{x\leq X} 1/x$ is roughly $\log X$. Thus, Tchébychev was vindicated and shown, using the appropriate measure, to be correct 99.59% of the time.
An obvious next step for Rubinstein and Sarnak was to apply their new counting model to other primes races. Staying close to our original example they showed that

$$\frac{1}{\log X} \sum_{\substack{x\leq X \\ \text{There are more primes } 3n+2 \\ \text{than } 3n+1 \text{ up to } x}} \frac{1}{x} \longrightarrow 0.9990...$$

Straying a little further but still remaining in well known territory we consider the race between $\mathrm{Li}(x)$ and $\pi(x)$. It was Gauss who first investigated this race and to this day there has not been found a point at which $\pi(x)$ takes the lead from $\mathrm{Li}(x)$. However, it is expected that this occurs at $x \approx 1.3982 \times 10^{316}$. This apparent sparsity of such events is reflected in the corresponding Rubinstein-Sarnak result.

$$\frac{1}{\log X} \sum_{\substack{x\leq X \\ \pi(x)<\mathrm{Li}(x)}} \frac{1}{x} \longrightarrow 0.99999973...$$

An assumption made by Rubinstein and Sarnak is that Riemann's stunning formulas such as (1.1) hold, which rely on the GRH being true. In addition they assume that the γ's appearing in these formulae are linearly independent over the rationals. As noted in [6],

> 'It would be shocking if either of these assumptions were false, but it does seem unlikely that either one will be proved in the near future.'

The work of Rubinstein and Sarnak has recently been improved by Daniel Fiorilli and and Greg Martin in a paper [4] (2013).

When considering prime races where primes are allocated to a team based on which arithmetic progression they lie in with respect to some modulus q, we have the following general phenomenon.

> 'If a is a nonsquare (mod q) and b is a square (mod q), then there tend to be more primes congruent to a mod q than b mod q in initial intervals of the positive integers; more succinctly, there is a tendency for $\pi(x;q;a)$ to exceed $\pi(x;q;b)$.' [4]

Rubinstein and Sarnak developed the notion of 'logarithmic density' in order to quantify such biases. We define $\delta(q;a,b)$ to be the logarithmic density of the set of numbers $x \geq 1$

satisfying $\pi(x;q,a) > \pi(x;q,b)$. Here, we use the logarithmic density of a set S of positive real numbers to be

$$\lim_{X\to\infty}\left(\frac{1}{\log X}\int_{\substack{1\le x\le X\\ x\in S}}\frac{dx}{x}\right).$$

We can think of $\delta(q;a,b)$ as the probability that $\pi(x;q,a) > \pi(x;q,b)$ when x is chosen at random. Fiorilli and Martin's great contribution to this area was to find a formula for $\delta(q;a,b)$ that allowed for accurate computations of these densities in hundreds of explicit cases. It is important to note that their work is based on the same assumptions as Rubinstein and Sarnak's, namely

1. The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet L-functions have real part equal to $1/2$.

2. The Linear Independence Hypothesis (LI): the imaginary parts of these non-trivial zeros are linearly independent over the rationals.

Their somewhat complicated formula is as follows.

Theorem 1.1.3. *Assume GRH and LI. Let q be a positive integer and $\rho(q)$ be the number of real characters* mod q. *Let a be a non-square residue* mod q *and b be a square residue* mod q. *Then for any non-negative integer K*

$$\delta(q;a,b) = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}}\sum_{l=0}^{K}\frac{1}{V(q;a,b)^l}\sum_{j=0}^{l}\rho(q)^{2j}s_{q;a,b}(l,j) + O_K\left(\frac{\rho(q)^{2K+3}}{V(q;a,b)^{K+3/2}}\right),$$

where the real numbers $s_{q;a,b}(l,j)$ are bounded in absolute value by a function of l uniformly in q, a, b and j.

Here $V(q;a,b)$ is defined as follows.

Definition 1.1.1 (Variance $V(q;a,b)$)**.** *For any reduced residues a and b (mod q), define*

$$V(q;a,b) = \sum_{\chi(mod\ q)}|\chi(b)-\chi(a)|^2 f(\chi),$$

where

$$f(\chi) = \sum_{\substack{\gamma\in\mathbb{R}\\ L(1/2+i\gamma,\chi)=0}}\frac{1}{1/4+\gamma^2}.$$

A series of subsequent results allow them to present the following data, which shows the 'top ten most unfair prime races'.

Theorem 1.1.4. *Assume GRH and LI. The ten largest values of $\delta(q;a,b)$ are given in Table 1.1.4.*

q	a	b	$\delta(q;a,b)$
24	5	1	0.999988
24	11	1	0.99983
12	11	1	0.999977
24	23	1	0.999889
24	7	1	0.999834
24	19	1	0.999719
8	3	1	0.999569
12	5	1	0.999206
24	17	1	0.999125
3	2	1	0.999063

Table 1.2: The top ten races with greatest values of $\delta(q;a,b)$.

In fact Fiorilli and Martin list all 117 prime races for which there is a bias greater than 9/10. Even more impressive is the accuracy with which this data has been computed. It is show that all error bounds obtained are at most 4.6×10^{-6}.

Such prime races, especially those involving primes in arithmetic progressions, are undeniably intriguing not least because of their unexpected nature. However, the reader might feel slightly uneasy that many of the results given above fail to provide a strong, or indeed, absolute description of these prime races. In the main body of this text we will explore one family of prime races for which we can give a far stronger commentary on how the races proceed and, in fact, name the winner(s) in each race.

Chapter 2

Big Biases

2.1 A Recent Result

For the remainder of this text we focus on one particular type of integer; those integers that have exactly two prime factors p and q of single multiplicities. In particular, for each real number x, we consider those integers $pq \leq x$ where p and q are prime. If we assume without loss of generality that $p \leq q$ then it is clear that all such numbers fall in to one of 4 'teams'. Either

1. $p \equiv q \equiv 1 \mod 4$,
2. $p \equiv 1 \mod 4$ and $q \equiv 3 \mod 4$,
3. $p \equiv q \equiv 3 \mod 4$
4. or $p \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$.

Thus, we have ourselves a prime race. Of further interest is the following data found by Dummit, Granville and Kisilevsky in their paper on *Big Biases Amongst Products Of Two Primes* [3]. For the quotient

$$r(x) := \frac{\#\{pq \leq x : \ p \equiv q \equiv 3 \mod 4\}}{\frac{1}{4}\#\{pq \leq x\}},$$

we have,

$$r(1000) \approx 1.347,\ r(10^4) \approx 1.258,\ r(10^5) \approx 1.212,\ r(10^6) \approx 1.183,\ r(10^7) \approx 1.162.$$

This shows a clear bias suggesting that Team 3 dominates. This race is easily generalised to a whole family of prime races by using the notion of a quadratic character. We look more closely at how these arise and how to construct them in Chapter 3 but the reader is

reminded that these are multiplicative, periodic functions with period q and take only the values 1 and -1.
One particular such character χ_{-4} can be defined by the *Kronecker* symbol $(-4/\cdot)$. When applied to primes, this function has the affect of distinguishing a prime p as follows.

$$\chi_{-4}(p) = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4, \\ -1 & \text{if } p \equiv 3 \mod 4. \end{cases}$$

We may then give the following equivalent definition for $r(x)$.

$$r(x) := \frac{\#\{pq \leq x : \ \chi_{-4}(p) = \chi_{-4}(q) = -1\}}{\frac{1}{4}\#\{pq \leq x\}}.$$

The use of quadratic characters in this way allows us to define a family of infinitely many races, each containing 4 teams. Given a quadratic character χ_d of conductor d (d is simply the absolute value of the modulus) we may consider the cases

1. $\chi_d(p) = \chi_d(q) = 1$,
2. $\chi_d(p) = 1$, and $\chi_d(q) = -1$,
3. $\chi_d(p) = \chi_d(q) = -1$,
4. and $\chi_d(p) = -1$ and $\chi_d(q) = 1$.

These cases define the four teams in each race. Quite remarkably, Dummit, Granville and Kisilevsky were able to show that in the χ_{-4} case there is a clearly defined winner. Moreover, assuming that $\mathcal{L}_\chi$ is non-zero for all quadratic characters χ (as is conjectured), then there is a winner in all such races. Furthermore, they gave a function that describes just how this lead behaves as x goes to infinity.

Theorem 2.1.1. *[Dummit, Granville and Kisilevsky, 2015] Let χ be a quadratic character of conductor d. For $\eta = -1$ or 1 we have*

$$\frac{\#\{pq \leq x : \ \chi(p) = \chi(q) = \eta\}}{\frac{1}{4}\#\{pq \leq x : \ (pq, d) = 1\}} = 1 + \eta\frac{(\mathcal{L}_\chi + o(1))}{\log\log x}, \quad \textit{where } \mathcal{L}_\chi := \sum_p \frac{\chi(p)}{p}.$$

It is conjectured in their paper that $\mathcal{L}_\chi \neq 0$ for all quadratic characters χ. This would imply that there is always such a bias. This result provides an answer to our original example in which we considered the primes mod 4 that make up $pq \leq x$. Taking $\chi = (-4/\cdot)$ and noting that $\mathcal{L}_\chi = -0.334...$ we have

$$r(x) \geq 1 + \frac{1 + o(1)}{3(\log\log x - 1)}.$$

Thus, for sufficiently large x, Team 3 dominates indefinitely in the mod 4 race.

We now give a complete proof of Theorem 2.1.1.

Proof. We may assume W.L.O.G that $p \le q \le x/p$ and $p \le \sqrt{x}$. We calculate,

$$\#\{pq \le x : \chi(p) = \chi(q) = 1\} = \sum_{\substack{p \le \sqrt{x} \\ \chi(p)=1}} \sum_{\substack{p \le q \le x/p \\ \chi(p)=1}} 1.$$

We use the Prime Number Theorem For Arithmetic Progressions (PNTAP's) (with a particular error term) in the form

$$\sum_{\substack{q \le x \\ \chi(q)=1 \\ q \text{ prime}}} 1 = \frac{x}{\phi(d)\log x} + O_d\left(\frac{x}{\phi(d)(\log x)^2}\right). \tag{2.1}$$

Using (2.1) we can evaluate the inner sum.

$$\begin{aligned}
\sum_{\substack{p \le q \le x/p \\ \chi(q)=1 \\ q \text{ prime}}} 1 &= \frac{x}{2p\log(x/p)} + O\left(\frac{x}{2p(\log(x/p))^2}\right) - \frac{p}{2\log p} + O\left(\frac{p}{2(\log p)^2}\right), \\
&= \frac{x}{2p\log(x/p)} + O\left(\frac{x}{4p(\log(x/p))^2} + \frac{p}{\log p}\right), \\
&= \frac{x}{2p\log(x/p)} + O\left(\frac{x}{p(2\log(x/p))^2} + \frac{p}{\log p}\right), \text{ since } p \le \sqrt{x} \\
&= \frac{x}{2p\log(x/p)} + O\left(\frac{x}{p(\log x)^2} + \frac{p}{\log p}\right).
\end{aligned}$$

So far we have

$$\#\{pq \le x;\, \chi(p) = \chi(q) = 1\} = \sum_{p \le \sqrt{x}} \left\{ \frac{\chi_0(p) + \chi(p)}{2} \cdot \frac{x}{2p\log(x/p)} + O\left(\frac{x}{p(\log x)^2} + \frac{p}{\log p}\right)\right\}.$$

We can use Merten's result: $\sum_{p \le x} \frac{1}{p} = \log\log x + O(1)$ to write

$$\begin{aligned}
\sum_{p \le \sqrt{x}} O\left(\frac{x}{p(\log x)^2}\right) &= O\left(\sum_{p \le \sqrt{x}} \frac{x}{p(\log x)^2}\right), \\
&= O\left(\frac{x}{(\log x)^2} \sum_{p \le \sqrt{x}} \frac{1}{p}\right), \\
&= O\left(\frac{x}{(\log x)^2} \log\log x\right).
\end{aligned} \tag{2.2}$$

Also, using partial summation, we have

$$\begin{aligned}\sum_{p\le\sqrt{x}}\frac{p}{\log p} &= O\left(\frac{\sqrt{x}}{\log x}\sum_{p\le\sqrt{x}}1\right),\\ &= O\left(\frac{\pi(\sqrt{x})\sqrt{x}}{\log x}\right),\\ &= O\left(\frac{x}{(\log x)^2}\right),\\ &= O\left(\frac{x}{(\log x)^2}\log\log x\right). \qquad (2.3)\end{aligned}$$

We note that $\chi(p)$ is non-zero only for $(p,d)=1$. In particular $\chi_0(p)$ is non-zero only for these values of p. Using results (2.2) and (2.3) we have

$$\#\{pq\le x;\,\chi(p)=\chi(q)=1\}=\frac{1}{4}\sum_{\substack{p\le\sqrt{x}\\(p,d)=1}}\frac{x}{p\log(x/p)}+\frac{x}{4}\sum_{p\le\sqrt{x}}\frac{\chi(p)}{p\log(x/p)}+O\left(\frac{x}{(\log x)^2}\log\log x\right).$$

The difference between the second sum and the same sum with $\log(x/p)$ replaced by $\log x$, is

$$\begin{aligned}\frac{x}{4}\sum_{p\le\sqrt{x}}\frac{\chi(p)}{p\log(x/p)}-\frac{x}{4}\sum_{p\le\sqrt{x}}\frac{\chi(p)}{p\log x} &= \frac{x}{4}\sum_{p\le\sqrt{x}}\left(\frac{\chi(p)\log x}{p\log x(\log x-\log p)}-\frac{\chi(p)(\log x-\log p)}{p\log x(\log x-\log p)}\right),\\ &= \frac{x}{4\log x}\sum_{p\le x}\frac{\chi(p)\log p}{p\log(x/p)}. \qquad (2.4)\end{aligned}$$

I claim that the summation in (2.4) is a member of the set $O\left(\frac{\log\log x}{\log x}\right)$. We estimate

$$\sum_{p\le\sqrt{x}}\frac{\chi(p)\log p}{p\log(x/p)}=\int_2^{\sqrt{x}}\frac{1}{t\log(x/t)}\,dS(t),\ \text{where } S(t)=\sum_{p\le t}\chi(p)\log p.$$

Now

$$\begin{aligned}S(t) &= \sum_{p\le t}\chi(p)\log p,\\ &= O\left(\log t\sum_{p\le t}\chi(p)\right),\\ &= O\left(\frac{t}{\log t}\right).\end{aligned}$$

Here we have used the result that there are, asymptotically, an equal number of $p \le t$ such that $\chi(p) = 1$ as there are with $\chi(p) = -1$. Then by (2.1) the above result follows.

By partial summation

$$\begin{aligned}
\sum_{p\le\sqrt{x}} \frac{\chi(p)\log p}{p\log(x/p)} &= \frac{S(\sqrt{x})}{\sqrt{x}\log\sqrt{x}} + \int_2^{\sqrt{x}} \frac{S(t)}{t^2\log(x/t)}\left(1+\frac{1}{\log(x/t)}\right)dt, \\
&= O\left(\frac{1}{(\log x)^2}\right) + O\left(\int_2^{\sqrt{x}} \frac{1}{t\log t\log(x/t)}\left(1+\frac{1}{\log(x/t)}\right)dt\right), \\
&= O\left(\frac{1}{(\log x)^2}\right) + O\left(\int_2^{\sqrt{x}} \frac{1}{t\log t\log(x/t)}\,dt\right), \\
&= O\left(\int_{\log 2}^{\log\sqrt{x}} \frac{1}{u(\log x - u)}\,du\right) + O\left(\frac{1}{(\log x)^2}\right), \text{ using the substitution } u = \log t \\
&= O\left(\int_{\log 2}^{\log\sqrt{x}} \frac{1}{\log x}\left(\frac{1}{\log x - u} + \frac{1}{u}\right)du\right) + O\left(\frac{1}{(\log x)^2}\right), \qquad (2.5) \\
&= O\left(\left[\frac{1}{\log x}(\log(\log x - u) + \log u)\right]_{\log x/2}^{\log\sqrt{x}}\right) + O\left(\frac{1}{(\log x)^2}\right), \\
&= O\left(\frac{\log\log x}{\log x}\right), \qquad (2.6)
\end{aligned}$$

as claimed, where we use partial fractions, choosing the constants appropriately to get (2.5).

Furthermore, I claim that $\sum_{p>\sqrt{x}} \frac{\chi(p)}{p} = O\left(\frac{1}{\log x}\right)$. Consider,

$$\begin{aligned}
\sum_{\sqrt{x}<p\le y} \frac{\chi(p)}{p} &= \sum_{\sqrt{x}<n\le y} \frac{\chi(p)(\pi(n) - \pi(n-1))}{n}, \\
&= \int_{\sqrt{x}}^{y} \frac{1}{t}\,dSt, \text{ where } S(t) = \sum_{p\le t}\chi(p) = O\left(\frac{t}{(\log t)^2}\right), \\
&= \left[\frac{S(t)}{t}\right]_{\sqrt{x}}^{y} + O\left(\int_{\sqrt{x}}^{y} \frac{1}{t(\log t)^2}\,dt\right), \\
&= O\left(\frac{1}{(\log y)^2} - \frac{1}{(\log x)^2}\right) + O\left(\left[(\log t)^{-1}\right]_{\sqrt{x}}^{y}\right), \\
&= O\left(\frac{1}{\log y} - \frac{1}{\log x}\right) \to O\left(\frac{1}{\log x}\right) \text{ as } y \to \infty,
\end{aligned}$$

as claimed. This gives us that

$$\frac{x}{4\log x}\sum_{p>\sqrt{x}}\frac{\chi(p)}{p} \ll \frac{x}{(\log x)^2}. \tag{2.7}$$

Using Results (2.6) and (2.7) we have

$$\#\{pq\le x;\,\chi(p)=\chi(q)=1\} = \frac{1}{4}\left\{\sum_{\substack{p\le\sqrt{x}\\(p,d)=1}}\frac{x}{p\log(x/p)}+\frac{x}{\log x}\sum_{p}\frac{\chi(p)}{p}+O\left(\frac{x}{(\log x)^2}\log\log x\right)\right\}.$$

I now claim that the first summation can be re-written as $\#\{pq\le x;\,(pq,d)=1\}$ with the incurred error being absorbed into 'O' term. To show this, we begin by computing the latter quantity. We know that

$$\#\{pq\le x;\,(pq,d)=1\} = \sum_{\substack{p\le\sqrt{x}\\(p,d)=1\\p\text{ prime}}}\sum_{\substack{p\le q\le x/p\\(q,d)=1\\q\text{ prime}}}1.$$

Noting that χ is a quadratic character, we may evaluate the inner sum using (2.1).

$$\begin{aligned}\sum_{\substack{p\le q\le x/p\\(q,d)=1\\q\text{ prime}}}1 &= \sum_{\substack{a \bmod d\\ \chi(a)=\pm1}}\sum_{\substack{q\equiv a \bmod d\\p\le q\le x/p\\q\text{ prime}}}1,\\ &= \phi(d)\left\{\frac{x}{\phi(d)p\log(x/p)}-\frac{p}{\phi(d)\log p}+o_d\left(\frac{x}{\phi(d)p\log(x/p)}-\frac{p}{\phi(d)\log p}\right)\right\},\\ &= \frac{x}{p\log(x/p)}-\frac{p}{\log p}+o_d\left(\frac{x}{p\log(x/p)}-\frac{p}{\log p}\right).\end{aligned}$$

Using (2.3) we know that each term after the first, when summed over $p\le\sqrt{x}$ with $(p,d)=1$, forms a member of $O\left(\frac{x}{(\log x)^2}\log\log x\right)$. This proves the previous claim and so

$$\#\{pq\le x;\,\chi(p)=\chi(q)=1\} = \frac{1}{4}\left\{\#\{pq\le x;\,(pq,d)=1\}+\frac{x}{\log x}\sum_{p}\frac{\chi(p)}{p}+O\left(\frac{x}{(\log x)^2}\log\log x\right)\right\}.$$

The first term is well known to equal $\frac{x}{\log x}(\log\log x+O(1))$. Dividing through by this term yields the desired result. The case where $\eta=-1$ is proved similarly. □

Chapter 3

Extreme Values Of $\mathcal{L}_\chi$

It is noted in [3], that our result (Theorem 2.1.1) gives a greater bias when we consider the character $(./5)$. In this case we have that $\mathcal{L}_{(./5)} \approx -1.008$. The data also confirms this more substantial bias.

$$r_5(1000) \approx 1.881,\ r_5(10^4) \approx 1.626,\ r_5(10^5) \approx 1.523,\ r_5(10^6) \approx 1.457.$$

Here, $r_d(x)$ denotes the analogous ratio where d is the modulus under inspection.

This poses two obvious questions.

(i) Can we find more extreme examples?

(ii) How large can this bias get?

The first we may answer directly. We start with a careful examination of real primitive characters.

3.0.1 Real Primitive Characters

We begin by examining the construction of a general Dirichlet character. The reader is reminded that the basic properties of characters ensure that they are periodic functions with modulus q that are completely multiplicative and defined on the integers co-prime to q. This definition extends naturally to all integers by defining the value of the function to be 0 on integers n with $(n, q) > 1$.
For a prime power modulus p^α, where $p \neq 2$, it is easily shown that our characters χ are given by

$$\chi(n) = \omega^{\nu(n)} \quad \text{for } (n, p) = 1,$$

where

$$\omega^{\phi(p^\alpha)} = 1.$$

Here $\nu(n)$ denotes the index of n relative to a fixed primitive root of the modulus p^α. For $q = 2^\alpha$ the situation is slightly more complicated, with the resulting characters given by

$$\chi(n) = \omega^{\nu(n)}(\omega')^{\nu'(n)},$$

where

$$\omega^2 = 1 \quad \text{and} \quad (\omega')^{2^{\alpha-2}} = 1.$$

Here, ν is defined to the modulus 2 and ω' is defined to the modulus $2^{\alpha-2} = \frac{1}{2}\phi(2^\alpha)$. Here we have used that every relatively prime residue class of the modulus 2^α can be uniquely represented as

$$(-1)^\nu 5^{\nu'}.$$

Using the fact that χ is completely multiplicative we know that, for a general modulus,

$$q = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} ...,$$

the general character can be written

$$\chi(n) = \chi(n; 2^\alpha)\chi(n; p_1^{\alpha_1})\chi(n; p_2^{\alpha_2})...$$

Characters can also be expressed in terms of the exponential function. As we have seen, the ω's integral to the construction of a character with modulus q are $\phi(q)$'th roots of unity. Thus we have

$$\omega = e^{2\pi i m/\phi(q)},$$

where different choices of ω arise from different choices of m, which are chosen from the $\phi(q)$ integers relatively prime to q.
Using this representation for the prime power modulus, we have that for p^α, $(p \neq 2)$

$$\chi(n; p^\alpha) = e^{2\pi i m\nu(n)/\phi(p^\alpha)}, \tag{3.1}$$

where, as before, $\nu(n)$ is the index of n relative to a particular primitive root of p^α. Similarly, for the 2^α modulus, we have

$$\chi(n; 2^\alpha) = e^{2\pi i m\nu(n)/2} \cdot e^{2\pi i m\nu'(n)/2^{\alpha-2}}, \tag{3.2}$$

where $n \equiv (-1)^\nu 5^{\nu'} \mod 2^\alpha$. Combining these we have, for the general modulus $q = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} ...$,

$$\chi(n) = exp\left(\frac{2\pi i m_0\nu_0}{2} + \frac{2\pi i m_0'\nu_0'}{2^{\alpha-2}} + \frac{2\pi i m_1\nu_1}{\phi(p_1^{\alpha_1})} + \frac{2\pi i m_2\nu_2}{\phi(p_2^{\alpha_2})} + ...\right), \tag{3.3}$$

where $(n, q) = 1$. In the formula above the integers m_0, m_0', m_1, $m_2, ...$ can take all values modulo the corresponding denominators.

From now on we deal only with primitive characters since it easily shown that all Dirichlet characters can be reduced to these. This is dealt with in [2] pg. 35-36.
Let us narrow our focus to real primitive characters; the kind under consideration in [3]. For which moduli do real primitive characters exist?

We begin by considering the prime power modulus p^α, $(p \neq 2)$. In this case we can write

$$\chi(n; p^\alpha) = e^{2\pi i m\nu(n)/p^{\alpha-1}(p-1)},$$

using (3.1) and the fact that $\phi(p^\alpha) = p^{\alpha-1}(p-1)$. The exponential function is only real if its argument is a multiple of π. Since the case when $\nu(n) = 1$ may indeed occur, we require that m be a multiple of $\frac{1}{2}p^{\alpha-1}(p-1)$. Furthermore, m is defined to be co-prime to p^α and so we must take $\alpha = 1$. The case where $m = 0$ simply gives the principal character. Finally, the condition that $m < p$ forces us to take $m = \frac{1}{2}(p-1)$. Thus

$$\chi(n; p^\alpha) = \chi(n, p) = e^{i\pi\nu(n)} = (-1)^{\nu(n)} = \left(\frac{n}{p}\right).$$

On the right hand side, the $\left(\frac{\cdot}{p}\right)$ is simply the Legendre symbol to the modulus p. This defines a primitive character. The reader is reminded of the formal definition of the Legendre symbol.

Definition 3.0.1 (Legendre symbol)**.** *For a fixed prime modulus p and a given integer n we define*

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv n \mod p \text{ has nonzero solutions,} \\ -1 & \text{if } x^2 \equiv n \mod p \text{ has no solutions,} \\ 0 & \text{if } p|n. \end{cases}$$

Now we deal with the modulus 2^α. If $\alpha = 1$ it is clear that we only get the principal character. For $\alpha = 2$ we can calculate explicitly the sole non-principal character directly and we find that

$$\chi_4(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 4, \\ -1 & \text{if } n \equiv -1 \mod 4. \end{cases}$$

which is clearly primitive since it has no smaller period than 4.
If $\alpha \geq 3$ then (3.2) gives us the general character

$$e^{2\pi i m\nu(n)/2} \cdot e^{2\pi i m\nu'(n)/2^{\alpha-2}},$$

where m, m', ν, ν' are defined as before. Again, we require that the argument of the exponentials here are both multiples of π. This implies that m' must be a multiple if $2^{\alpha-3}$. For $\alpha > 3$, m' is not co-prime to the modulus and so our character is not primitive. Thus the only remaining case is when $\alpha = 3$.

This leaves us with three sub-cases.

(i) $m_0 = 1$, $m_0' = 0$.

(ii) $m_0 = 1$, $m_0' = 1$.

(iii) $m_0 = 1$, $m_0' = 0$.

The last of these cases gives us χ_4 again but to the modulus 8. This is imprimitive since it has a lesser modulus of 4 and so we discard it. We may calculate directly the structure of the characters in the remaining two cases. The first gives us,

$$\chi_8(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \mod 8, \\ -1 & \text{if } n \equiv \pm 3 \mod 8, \end{cases}$$

and the second gives us $\chi_8(n)\chi_4(n)$. It is easy to check that these are both primitive.

Thus we have found all moduli for which there exist real primitive characters. They are

1. p (> 2) with the character $(n|p)$,
2. 4 with the character χ_4,
3. 8 with the characters χ_8, and $\chi_4\chi_8$.

It would be nice if we were able to express these basic characters in terms of the Kronecker symbol for ease of working with them later on.

3.0.2 The Kronecker Symbol

For the sake of linearity we begin by recalling the definition of the Legendre symbol.

Definition 3.0.2 (Legendre symbol)**.** *For a fixed prime modulus p and a given integer n we define*

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \textit{if } x^2 \equiv n \mod p \textit{ has nonzero solutions}, \\ -1 & \textit{if } x^2 \equiv n \mod p \textit{ has no solutions}, \\ 0 & \textit{if } p|n. \end{cases}$$

This definition can be easily extended to to $(n|b)$ where b is any odd integer. If b is simply an odd prime then $(n|b)$ is just the Legendre symbol. If b is composite and $b = p_1^{e_1}p_2^{e_2}p_3^{e_3}\cdot\ldots$ then we may extend by multiplicity, setting

$$\left(\frac{n}{b}\right) = \left(\frac{n}{p_1}\right)^{e_1}\left(\frac{n}{p_2}\right)^{e_2}\left(\frac{n}{p_3}\right)^{e_3}\cdot\ldots$$

Odd integers are easily accounted for my setting $(\frac{n}{-1}) = sgn(n)$. We call this extension of the Legendre symbol the *Jacobi symbol*. By the law of quadratic reciprocity we have

$$\left(\frac{n}{b}\right) = (-1)^{(n-1)(b-1)/4}\left(\frac{b}{n}\right),$$

provided that n is odd. This allows us to think of n as being fixed and b allowed to vary. As discussed in [5], it is sensible to define

$$\left(\frac{n}{2}\right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \mod 8, \\ -1 & \text{if } n \equiv \pm 3 \mod 8, \\ 0 & \text{if } 2|n. \end{cases}$$

Finally, we have defined the *Kronecker symbol* $(b|n)$ for all integers b and n. We can now express all of the basic characters in terms of quadratic residue symbols. They are

$$\left(\frac{-4}{n}\right), \left(\frac{8}{n}\right), \left(\frac{-8}{n}\right), \left(\frac{p}{n}\right).$$

Here the modulus of each character is the absolute value of the top number. Therefore the set of all real primitive characters is exactly the set of symbols $(d|n)$ where d is the product of relatively prime factors of the form

$$-4, \quad 8, \quad -8, \quad (-1)^{\frac{1}{2}(p-1)}p \ \ (p > 2);$$

and each corresponding character has modulus $|d|$. Each number of the form $(-1)^{\frac{1}{2}(p-1)}p$ is congruent to 1 mod 4 and products of distinct factors of this form generate all square-free integers that are congruent to 1 mod 4. If we multiply these numbers by -4 we get all integers of the form $4N$ where N is square free and congruent to 3 mod 4. Also we get all such numbers multiplied by ± 8 which gives all integers of the form $4N$ where N is congruent to 2 mod 4. This leaves us with the following.
Every primitive real character can be expressed using the Kronecker symbol $(d|n)$ *where* d *is*

(i) a square-free integer congruent to 1 mod 4 or,

(ii) an integer of the form $4N$ where N is square-free and congruent to 2 or 3 mod 4.

3.0.3 Computations

This classification of the real primitive characters allows us to search, much more easily, for more extreme values of $\mathcal{L}_\chi$.
The reader will note that (5/.) (using the Kronecker symbol) falls into case (i). This gave

a value of $\mathcal{L}_{(./5)} \approx -1.008$.
In looking for more extreme examples it is clear that one must calculate the value of $\mathcal{L}_\chi$ for many different quadratic characters χ. In the previous section we developed an insight, which will greatly speed up this process. We can begin by generating a list of numbers up to, say, 500000; each number having one of the two forms above. This list will then contain all the numbers (up to 500000) corresponding to the conductors of real primitive characters. Then all that remains is to calculate the sum $\mathcal{L}_{(d/.)}$ for each d in our list.

To implement this search I used the SageMath programming language since is has an inbuilt

```
kronecker(x,y)
```

function that made writing the program straightforward. Another advantage was the ability to use the SageMathCloud project which, as the name suggests, provides a cloud computing platform where I was able to leave the process running for 24 hours without attendance. See Appendix A for the Sage code used.

As expected, this very quickly gave some more substantial biases. The first case worth mentioning is the character χ given by

$$\chi(n) = \left(\frac{-163}{n}\right).$$

This gives the significantly more negative value of $\mathcal{L}_\chi \approx -1.5842$. The values of $\mathcal{L}_\chi$ given in this document have been computed using the first 10 million primes. It is noted in [3] that we can write

$$\mathcal{L}_\chi = \log L(1,\chi) + E(\chi), \text{ where } -0.315718... \leq E(\chi) \leq -0.18198.... \tag{3.4}$$

It is then clear that $\mathcal{L}_\chi$ is rapidly converging and the level of accuracy given using the first 10 million primes is more than adequate. In order to quantify the possible error in only

computing the truncated sum, consider

$$\begin{aligned}
\mathcal{L}_\chi - \sum_{\substack{p\le x\\ p \text{ prime}}} \frac{\chi(p)}{p} &= \sum_{\substack{p>x\\ p \text{ prime}}} \frac{\chi(p)}{p},\\
&= \int_x^\infty \frac{dS(t)}{t}, \text{ where } S(t) \ll t^{\frac{1}{2}}\log(dt) \text{ under GRH}\\
&= \left[\frac{S(t)}{t}\right]_x^\infty + \int_x^\infty \frac{S(t)}{t^2}\,dt\\
&\ll \frac{\log(dx)}{\sqrt{x}} + \int_x^\infty \frac{\log(dt)}{t^{\frac{3}{2}}}\,dt\\
&\ll \frac{\log(dx)}{\sqrt{x}} + \left[\frac{-2\log(dt)}{\sqrt{t}}\right]_x^\infty + 2\int_x^\infty \frac{1}{t^{\frac{3}{2}}}\,dt\\
&\ll \frac{\log(dx)}{\sqrt{x}} + \frac{1}{\sqrt{x}},\\
&\ll \frac{\log(dx)}{\sqrt{x}}. \qquad (3.5)
\end{aligned}$$

Let us, for a moment, suppose that the implicit constant in the 'big-O' notation above is 1. Then we may use (3.5) to give an explicit value for the error in our computing of $\mathcal{L}_\chi$. We sum over the first 10000000 primes, which implies that the maximum error in computing $\mathcal{L}_\chi$, where the character χ has modulus q, is

$$E(\mathcal{L}_\chi, q) = \frac{\log(179424691|q|)}{\sqrt{179424691}}. \qquad (3.6)$$

The large integer here is simply (prime #10000001) -1. It is interesting to note that that if we did know the constant in (3.5), this would greatly help to advance the search for a new most extreme value of $\mathcal{L}_\chi$ using the method outlined in Appendix A. The bottleneck for the computation is in computing a sum of 10^7 terms. Knowing that reducing the number of terms here would only effect the computed value by a given amount would assure us that new extreme values were indeed representative of a more extreme true value. Looking back at our calculation of $\mathcal{L}_{(-163/\cdot)}$ we can use (3.6) to say (assuming the constant to be 1 and GRH) that

$$E(\mathcal{L}_\chi, -163) \approx 0.0018$$

If we define the analogous quantity to $r(x)$ used in [3] by

$$r_q(x) = \frac{\#\{pq \le x :\ \chi_q(p) = \chi_q(q) = \eta\}}{\frac{1}{4}\#\{pq \le x :\ (pq, |q|) = 1\}},$$

we are able to calculate the real data corresponding to each character χ. We have that

$$r_{-163}(10^3) \approx 2.470,\ r_{-163}(10^4) \approx 2.020,\ r_{-163}(10^5) \approx 1.793,\ r_{-163}(10^6) \approx 1.687.$$

This data shows, as expected, that there is an even greater bias in the case of $\chi(n) = \left(\frac{-163}{n}\right)$.

Continuing the search for more extreme examples, we find that $q = 5417453$ gives $\mathcal{L}_\chi \approx -1.8452$. Again the data confirms this bias.

$$r_{5417453}(10^3) \approx 3.134,\ r_{5417453}(10^4) \approx 2.383,\ r_{5417453}(10^5) \approx 2.026,\ r_{5417453}(10^6) \approx 1.860.$$

We may also use (3.6) to give the maximum possible error in the calculation of $\mathcal{L}_\chi$ in this case.

$$E(\mathcal{L}_\chi, 5417453) \approx 0.0026,$$

3.0.4 A Shortcut

Trawling through potential moduli in this way and calculating $\mathcal{L}_\chi$ for each one is a very direct but by no means an optimal way of finding these extreme values that we seek. We have from (3.4) that computing $\mathcal{L}_\chi$ is equivalent to computing

$$\log L(1, \chi) + E(\chi).$$

We know that $E(\chi)$ converges rapidly and is bounded as in (3.4). Therefore, if we can find very small values of $L(1, \chi)$, close to 0, then the corresponding modulus should give us some interesting values for $\mathcal{L}_\chi$. It so happens that the value of L-functions at 1 has been the topic of much research over the years. In 1970 Derrick H. Lehmer, Emma Lehmer and Daniel Shanks produced a paper [1] in which they compute some exceedingly small values for $L(1, \chi)$. The smallest values that they found is

$$L(1, \chi) = 0.169(5),$$

where the corresponding modulus is 49107823133. Indeed, this modulus gives the most extreme value of $\mathcal{L}_\chi$ of any in the paper and also of this text thus far.

$$\mathcal{L}_\chi \approx -1.9557.$$

The error in this calculation, using the argument at the end of Section 3.0.1 is

$$E(\mathcal{L}_\chi, 49107823133) \approx 0.0033.$$

If one was to use the computational methods employed to compute the data in [1] on modern day computers I would expect the results to be very fruitful in the context of our search.[1]

[1] I believe the most efficient way to compute such data currently would be to use the ComputeL pari package developed by Tim Dokchitser. This is designed to compute special values of L-functions and their derivatives numerically. [8].

Clearly one could go on in this manner, looking for greater and greater biases but it is far more interesting and indeed fruitful to understand why such, seemingly random, numbers yield the extreme biases that they do. Let us examine the number -163.
We are interested in the most negative values of $\mathcal{L}_\chi$ that we can find. This is clearly when, for a given modulus q, $(q/p) = -1$ as often as possible (ranging over p). Further more, we want this high density of -1 values to occur early on in the sum since the first few terms hold the highest weight in determining its overall value.

We now examine a much faster way of finding such moduli.

3.0.5 Prime Generating Polynomials

It was Euler in 1772 who first came across what became 'Euler's Polynomial'. It is the unassuming quadratic

$$x^2 + x + 41.$$

What is so remarkable is that for the integers $0 \leq x \leq 39$, the value of $x^2 + x + 41$ is prime. For $x = 40$ it is clear that the value of this quadratic is no longer prime. If we now consider the general quadratic

$$f(x) = ax^2 + bx + c,$$

then, for a given integer x, the value of this quadratic is divisible by a prime p if and only if

$$ax^2 + bx + c \equiv 0 \mod p,$$

which is equivalent to saying that

$$(2ax + b)^2 \equiv b^2 - 4ac \mod p.$$

Said differently $\left(\frac{\Delta(f)}{p}\right) = 1$. This suggests that if a given quadratic produces a high density of primes for integers $x \leq m$, then the character corresponding to $\chi(n) = \left(\frac{\Delta(f)}{n}\right)$ might produce a high density of -1 values for $n \leq m$. Now, we have that $\Delta(x^2 + x + 41) = -163$! So it is due directly to the high number of primes produced by Euler's Polynomial that our character $(-163/n)$ gives us a particularly low value of $\mathcal{L}_\chi$.
Similarly, constructing a quadratic with discriminant 5417453, say

$$x^2 + x - 1354363,$$

we find that this produces a prime roughly 34% of the time over the first 1 million natural numbers. For the reader's reference, the corresponding value for Euler's Polynomial is 26%. This connection would suggest that, in looking for extreme values of $\mathcal{L}_\chi$, we would do well

to find quadratics that yield primes for a high density of input values x less than some sensible bound m. As has already been noted, it is more than sufficient to take the first 10 million terms in the computation of $\mathcal{L}_\chi$. With this in mind we will now look for quadratics that take prime values for a high density of input integers $0 \leq x \leq m$ where we take $m = 10,000,000$. Looking more carefully, if two quadratics have roughly the same density of prime values for $x \leq m$, then it will be the quadratic whose density of prime values is skewed towards those produced by lower values of x that will be of more interest to us. It is therefore instructive to look for polynomials with high asymptotic prime values but not necessarily essential in finding the 'best' moduli for low values of $\mathcal{L}_\chi$.

3.0.6 Jacobson And Williams 2002

We now turn to an article by Michael J. Jacobson and Hugh C. Williams published in 2002 [9]. This work is based upon a conjecture of Hardy and Littlewood, namely 'Conjecture F'. This implies that, for a polynomial of the form $f(x) = x^2 + x + A$, $A \in \mathbb{Z}$ with discriminant Δ, the asymptotic density of prime values of f is related to a quantity $C(\Delta)$. They also suggest that the larger the values of $C(\Delta)$, the higher the asymptotic density of primes for *any* polynomial of discriminant Δ. We thus restrict to polynomials of the form $f(x) = x^2 + x + A$. If we also denote by $P_A(n)$, the number of primes taken by $f_A(x)$ for $0 \leq x \leq n$ then Conjecture F can be written as follows.

Conjecture 3.0.1 (Conjecture F (simplified), 1923)**.**

$$P_A(n) \sim C(\Delta) L_A(n),$$

where

$$L_A(n) = 2 \int_0^n \frac{dx}{\log f_A(x)},$$

and

$$C(\Delta) = \prod_{p \geq 3} 1 - \frac{\left(\frac{\Delta}{p}\right)}{p-1}.$$

Clearly, it is the value of $C(\Delta)$ which is the determining quantity in the asymptotic density of primes assumed by f_A. Once again, we see that there is a dependency on the distribution of values of (Δ/p) with a high density of -1 values, giving rise to a large value of $C(\Delta)$. Jacobson and Williams state that the largest value of $C(\Delta)$ known before their work was

$$C(-13598858514212472187) = 5.3670819.$$

They also make the following remarks.

> 'The corresponding polynomial $x^2+x+3399714628553118047$ starts off slower than Euler's polynomial (only 24 primes for $x \leq 100$ compared to 87), but for $x \leq 10^7$ it assumes 2517022 prime values as compared to only 22081974 by Euler's polynomial. Notice that for Euler's polynomial we have $C(-163) = 3.3197732$, so by Conjecture F we expect that it will assume fewer prime values asymptotically than $x^2+x+3399714628553118047$.

As the value of Δ gets larger, calculating the actual density of primes assumed by the corresponding polynomial becomes more difficult and beyond a certain point (with current computing power) infeasible. However, only assuming the Extended Riemann Hypothesis (ERH) Jacobson and Williams developed a method for calculating larger values of $C(\Delta)$ than had previously been seen. The largest value of $C(\Delta)$ that they found was $C(\Delta) = 5.65726388$. It is thus expected that the corresponding polynomial, x^2+x+A 'has the highest asymptotic density of prime values for any polynomial of this type currently know', where A is given by

$$-33251810980696878103150085257129508857312847751498190349983874538507313.$$

We would thus expect the character given by $\chi(n) = (d/n)$ where $d = 1-4A$ to yield a low value of $\mathcal{L}_\chi$. Firstly, we note that $d \equiv 1 \mod 4$, square-free and thus (d/n) defines a real primitive character. A simple piece of SageMath yields

$$\mathcal{L}_{(d/.)} \approx -2.1108.$$

This is by far the most extreme value of $\mathcal{L}_\chi$ that we have seen. Furthermore, the data confirms this bias.

$$r_d(10^3) \approx 3.847,\ \ r_d(10^4) \approx 2.974,\ \ r_d(10^5) \approx 2.394,\ \ r_d(10^6) \approx 2.067$$

As we touched on earlier, there may indeed be polynomials with values of $C(\Delta)$ that assume smaller asymptotic density of prime values but give rise to more extreme values of $\mathcal{L}_\chi$ since they may have better 'low value' densities than the polynomial above.

Chapter 4

How Large Can This Bias Get?

In the previous chapter we saw that the bias under consideration can become really quite large. That is, for the range of q over which we can compute $\mathcal{L}_{\chi_q}$, we have seen some very extreme values. Since our general result tells us that the bias overall will diminish towards vanishing at infinity, it is interesting to ask the following.

$$\text{How large can } r_d(x) \text{ get if } d \leq x?$$

A potential answer to this question is suggested by the following conjecture [3].

Conjecture 4.0.1 (Dummit, Granville and Kisilevsky, 2015)**.** *There exists $d \leq x$ such that $r_d(x)$ is as large as*

$$1 + \frac{\log\log\log x + O(1)}{\log\log x}.$$

In this chapter we first explore how this might be proven assuming the GRH. This will give us an essential feel for the problem. We will then show how this Conjecture can indeed be proven unconditionally and we give a proof of this.

4.1 Assuming GRH

Let d be the conductor of χ and let us search over $d \leq x$ for a large bias. We begin

$$\begin{aligned}\frac{\#\{ab \leq x;\ \chi(a) = \chi(b) = 1\}}{\frac{1}{4}\#\{ab \leq x;\ (ab, d) = 1\}} &= \sum_{\substack{ab \leq x \\ a,b \text{ prime}}} (1 + \chi(a))(1 + \chi(b)) \Big/ \sum_{\substack{ab \leq x \\ a,b \text{ prime}}} 1, \\ &= 1 + \sum_{\substack{ab \leq x \\ a,b \text{ prime}}} \big(\chi(a) + \chi(b) + \chi(ab)\big) \Big/ \sum_{\substack{ab \leq x \\ a,b \text{ prime}}} 1. \qquad (4.1)\end{aligned}$$

Now, by Landau

$$\sum_{\substack{ab\leq x\\ a,b \text{ prime}}} 1 = \frac{x}{\log x}\left(\log\log x + O(1)\right).$$

Then (4.1) becomes

$$= 1 + \frac{\log x}{x(\log\log x + O(1))} \sum_{\substack{ab\leq x\\ a,b \text{ prime}}} (\chi(a) + \chi(b) + \chi(ab)).$$

We can then examine

$$\sum_{\substack{ab\leq x\\ a,b \text{ prime}}} \chi(a) = \sum_{\substack{a\leq\sqrt{x}\\ a \text{ prime}}} \chi(a) \sum_{\substack{b\leq x/a\\ b \text{ prime}}} 1 + \sum_{\substack{b\leq\sqrt{x}\\ b \text{ prime}}} 1 \sum_{\substack{a\leq x/b\\ a \text{ prime}}} \chi(a) - \sum_{\substack{a\leq\sqrt{x}\\ a \text{ prime}}} \chi(a) \cdot \sum_{\substack{b\leq\sqrt{x}\\ b \text{ prime}}} 1. \tag{4.2}$$

To get a useful bound for the terms in this expression it is clearly necessary to understand the quantity

$$\sum_{\substack{n\leq x\\ n \text{ prime}}} \chi(n).$$

We begin

$$\begin{aligned}
\sum_{\substack{n\leq x\\ n \text{ prime}}} \chi(n) &= \sum_{n\leq x} \frac{\chi(n)\Lambda(n)}{\log n},\\
&= \int_2^x \frac{1}{\log t}\, dS(t), \text{ where } S(t) = \sum_{n\leq t} \chi(n)\Lambda(n),\\
&= \left[\frac{S(t)}{\log t}\right]_2^x + \int_2^x \frac{S(t)}{t(\log t)^2}\, dt.
\end{aligned}$$

Now, under the GRH, we have, from [2] (pg. 125), that

$$S(t) \ll t^{1/2}(\log dt)^2.$$

Using this gives that

$$\begin{aligned}
\sum_{\substack{n\leq x\\ n \text{ prime}}} \chi(n) &\ll \left[t^{1/2}\frac{(\log(dt))^2}{\log t}\right]_2^x + \int_2^x \frac{(\log(dt))^2}{t^{1/2}(\log t)^2}\, dt,\\
&\ll x^{1/2}(\log(dx)) + x^{1/2},\\
&\ll x^{1/2}(\log x) \text{ for } d \leq x.
\end{aligned} \tag{4.3}$$

Using (4.3), and examining the second term in (4.2) we have

$$\begin{aligned}\sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} 1 \sum_{\substack{a\le x/b\\ a \text{ prime}}} \chi(a) &\ll \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \sqrt{x/b}\log\left(\frac{x}{b}\right),\\ &\ll x^{1/2}\log x \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \frac{1}{b^{1/2}},\\ &\ll x^{3/4}\log x, \end{aligned} \tag{4.4}$$

where we have used that

$$\begin{aligned}\sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \frac{1}{b^{1/2}} &\ll 2+\int_1^{\sqrt{x}} \frac{1}{t^{1/2}}\,dt,\\ &= 2+\left[2b^{1/2}\right]_1^{\sqrt{x}},\\ &\ll x^{1/4}.\end{aligned}$$

Applying (4.3) to the third term in (4.2) we have that

$$\begin{aligned}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a)\cdot \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} 1 &\ll x^{1/4}\log x\cdot\frac{\sqrt{x}}{\log x},\\ &\ll x^{3/4}. \end{aligned} \tag{4.5}$$

The first term in (4.2) is a little different. We have,

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \sum_{\substack{b\le x/a\\ b \text{ prime}}} 1 = \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a)Li(x/a) + O\left(x^{3/4}\log x\right),$$

where we have used (4.3) and the following result due to Helge Von Koch (1901) that assumes the Riemann Hypothesis (RH).

$$\pi(x) = Li(x) + O(\sqrt{x}\log x).$$

We get the same results for $\sum_{\substack{ab\le x\\ a,b \text{ prime}}} \chi(b)$ in (4.1), with two terms giving a small error term and the last giving a sum and error term as above. Finally the third term in (4.1) may be written as

$$\sum_{\substack{ab\le x\\ a,b \text{ prime}}} \chi(ab) = \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \sum_{\substack{b\le x/a\\ a \text{ prime}}} \chi(b) + \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \chi(b) \sum_{\substack{a\le x/b\\ a \text{ prime}}} \chi(a) - \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a)\cdot \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \chi(b).$$

The first term here can be manipulated, using (4.3), as follows.

$$\begin{aligned}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\chi(a)\sum_{\substack{b\le x/a\\ b \text{ prime}}}\chi(b) &\ll \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\chi(a)\sqrt{x/a}\log x/a,\\ &\ll x^{1/2}\log x\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\frac{1}{a^{1/2}},\\ &\ll x^{3/4}\log x.\end{aligned}$$

Similarly, by interchanging a and b, the second term has the same bound. Finally

$$\begin{aligned}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\chi(a)\cdot\sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}}\chi(b) &\ll x^{1/4}\log x\cdot x^{1/4}\log x,\\ &\ll x^{3/4}\log x.\end{aligned}$$

The result of all of this is the following.

$$\begin{aligned}\frac{\#\{ab\le x;\ \chi(a)=\chi(b)=1\}}{\frac{1}{4}\#\{ab\le x;\ (ab,d)=1\}} &= 1+\frac{\log x}{x\log\log x}\left(\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\chi(a)Li(x/a)+O\left(x^{3/4}\log x\right)\right),\\ &= 1+\frac{\log x}{x\log\log x}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\chi(a)Li(x/a)+O\left(\frac{(\log x)^2}{x^{1/4}\log\log x}\right),\end{aligned}$$

where we have been careful to note that although two terms contributed to the sum here, we only need include one since until now we have been using $ab\le x$, which allows for interchanging a and b. Our quantity on the LHS, however, requires that one is less than or equal to the other.
Now let us examine

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\chi(a)Li(x/a)=\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\frac{x\chi(a)}{a\log(x/a)}+O\left(x\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}}\frac{1}{a(\log(x/a))^2}\right),$$

where we have used that

$$\chi(a)Li(t)=\frac{\chi(a)t}{\log t}+O\left(\frac{t}{(\log t)^2}\right).$$

It is easy to show that the last term is $O(x/\log x)$.

$$\begin{aligned}\left| x \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a(\log(x/a))^2} \right| &\ll x \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{1}{a(\log(x/a))^2}, \text{ using } |\chi(a)| \le 1 \\ &\ll \frac{x}{(\log x)^2} \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{1}{a}, \\ &\ll \frac{x \log\log x}{(\log x)^2} = o\left(\frac{x}{\log x}\right),\end{aligned} \tag{4.6}$$

where we have used the following result due to Merten.

$$\sum_{\substack{a\le x\\ a \text{ prime}}} \frac{1}{a} = \log\log x + O(1).$$

We now aim to change the summand of the first term into something we can deal with and in so doing create an error term that is no bigger than the current $O(x/\log x)$. Consider

$$\begin{aligned}\left| \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{x\chi(a)}{a\log x} - \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{x\chi(a)}{a\log(x/a)} \right| &= x \left| \frac{1}{\log x} \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)(\log x - \log a - \log x)}{a\log(x/a)} \right|, \\ &\le \frac{x}{\log x} \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{\log a}{a\log x/a}, \\ &\le \frac{2x}{(\log x)^2} \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{\log a}{a}, \\ &\le \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),\end{aligned} \tag{4.7}$$

by the result of Merten that

$$\sum_{\substack{p\le x\\ p \text{ prime}}} \frac{\log p}{p} = \log x + O(1).$$

The result of this is that

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) Li(x/a) = \frac{x}{\log x} \sum_{a \text{ prime}} \frac{\chi(a)}{a} + \frac{x}{\log x} \sum_{\substack{a>\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a} + O\left(\frac{x}{\log x}\right).$$

Now

$$\sum_{\substack{a>\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a} \ll \frac{\log(dx)}{x^{1/4}}, \text{ by (3.5)}$$

$$\ll \frac{\log x}{x^{1/4}},$$

for $d \leq x$. Finally we arrive at the following formula.

$$\frac{\#\{ab \leq x;\ \chi(a) = \chi(b) = 1\}}{\frac{1}{4}\#\{ab \leq x;\ (ab, d) = 1\}} = 1 + \frac{\mathcal{L}_\chi + O(1)}{\log\log x}, \quad d \leq x. \tag{4.8}$$

If we can show that for sufficiently large x, we can find $d \leq x$ such that

$$\mathcal{L}_\chi = \log\log\log x + O(1),$$

then we have proved the Conjecture assuming the GRH.

4.2 An Unconditional Result

In §14 of [2] we have the following Theorem due to Gronwall and Titchmarsh. We will keep the labelling of constants consistent with that seen in the text for the reference of the reader.

Theorem 4.2.1. *There exists a positive constant c_{14} with the following property. If χ is a complex character modulo q, then $L(s, \chi)$ has no zero in the region defined by*

$$\sigma \geq \begin{cases} 1 - \frac{c_{14}}{\log q|t|} & \text{if } |t| \geq 1, \\ \\ 1 - \frac{c_{14}}{\log q} & \text{if } |t| \leq 1. \end{cases} \tag{4.9}$$

If χ is a real non-principal character then the only possible zero of $L(s, \chi)$ in this region is a single (simple) real zero.

It is interesting to note that a value for c_{14} was computed by H. Kadiri in 2005 [11]. Her result is as follows.

Theorem 4.2.2. *Let $\mathcal{L}_q(s)$ be the products of Dirichlet L-functions modulo q. Then $\mathcal{L}_q(s)$ has at most one simple zero in the region*

$$\sigma \geq 1 - \frac{1}{R_0 \log(\max(q, q|t|))}, \text{ where } R_0 = 6.3970.$$

Such a zero, if it exists, is real, simple and corresponds to a real non-principal character modulo q. We shall refer to it as an exceptional zero and q as an exceptional modulus.

We are only interested in the case when χ is a non-principal real character. In the same Chapter, it is noted that Page went on to deduce the following crucial Theorem where we consider positive integers $q \leq x$ where $x \geq 3$. This Theorem isolates the implication of the previous result for a real non-principal character χ with modulus q.

Theorem 4.2.3. *If c_{20} is a suitable positive constant, there is at most one real primitive χ to a modulus $q \leq x$ for which $L(s,\chi)$ has a real zero β satisfying*

$$\beta > 1 - \frac{c_{20}}{\log x}. \tag{4.10}$$

I now outline the important steps in giving an explicit formula that describes the behaviour of $\psi(x,\chi)$ as presented in [2].
In section §19, of [2], a rather technical argument involving contour integration is used to arrive at the following formula for $\psi(x,\chi)$.

$$\psi(x,\chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} + \sum_{|\gamma|<1} \frac{1}{\rho} + R_2(x,T), \tag{4.11}$$

where

$$R_2(x,T) \ll xT^{-1}\log^2(qx).$$

The sums here are over the non-trivial zeros ρ of $L(s,\chi)$. Also $\gamma = Im(\rho)$ and $2 \leq T \leq x$ is a parameter incurred in the contour integration that went before. If we now consider only real χ, then we know that there may be at most one simple real zero β_1 of $L(s,\chi)$ such that

$$|\gamma| < 1, \quad \beta_1 > 1 - \frac{c}{\log q}. \tag{4.12}$$

Here c is a numerical constant, which we can suppose is less than $\frac{1}{4}$. By the symmetry of $L(s,\chi)$ there will also be a zero at $1-\beta_1$ (if indeed β_1 exists). We then partition the sums in (4.11) appropriately so as to single out these 'exceptional' zeros. Writing $\sum'$ to denote the sum over non-trivial zeroes excluding the exceptional ones, we have

$$\psi(x,\chi) = -\sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + \sum_{|\gamma|<1}{}' \frac{1}{\rho} - \frac{x^{\beta_1}-1}{\beta_1} - \frac{x^{1-\beta_1}-1}{1-\beta_1} + R_2(x,T).$$

From (4.12) we can see that, for the zeros under consideration in the second sum,

$$\rho^{-1} = O(\log q). \tag{4.13}$$

Furthermore, we have the following proposition from §16 of [2].

Proposition 4.2.1. *If χ is a primitive character to the modulus q and $N(T,\chi)$ denotes the number of zeroes of $L(x,\chi)$ in the rectangle*

$$0 < \sigma < 1, \quad |t| < T,$$

then we have

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi} - \frac{T}{\pi} + O(\log T + \log q).$$

Applying this result to the zeroes of the second sum in our expression we have that there are $O(\log q)$ terms here. This is achieved by taking $T = 2$. This allows the second sum to be absorbed into the error term by (4.13).
The β^{-1} term can be dropped since it is clearly $O(1)$. Finally we deal with the $1-\beta_1$ term by considering two cases:

1. $(1-\beta_1)\log x < 1$,
2. $(1-\beta_1)\log x > 1$.

In the first case we have

$$\begin{aligned}\frac{x^{1-\beta_1}-1}{1-\beta_1} &= \frac{1}{1-\beta_1}\sum_{i=1}^{\infty}((1-\beta_1)\log x)^i,\\ &\ll \frac{(1-\beta_1)\log x}{1-\beta_1}, \quad \text{since } (1-\beta_1)\log x < 1\\ &\ll \log x,\end{aligned}$$

where we have used the series expansion of the exponential function. The second case can be dealt with as follows.

$$\begin{aligned}\frac{x^{1-\beta_1}-1}{1-\beta_1} &\ll x^{1-\beta_1}\log x + \log x,\\ &\ll x^{\frac{1}{4}}\log x,\end{aligned}$$

where we have used that $\beta_1 > \frac{3}{4}$ by (4.12), as $q \geq 3$. Thus we arrive at the following expression for $\psi(x,\chi)$, valid for primitive χ and $2 \leq T \leq x$. In [2] this is easily shown to hold for all non-principal characters χ.

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} - \sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + R_3(s,T), \tag{4.14}$$

where

$$|R_3(x,T)| \ll xT^{-1}\log^2(qx) + x^{\frac{1}{4}}\log x.$$

The term $-x^{\beta_1}/\beta_1$ can be omitted unless χ is a real character for which $L(s,\chi)$ has a zero β_1 (which is necessarily unique and simple) satisfying

$$\beta_1 > 1 - \frac{c}{\log q},$$

where c is a positive constant (absolute). We reiterate here that there is at most one real character (mod q) for which such a zero can exist.
Every zero in the sum of (4.14) trivially satisfies

$$\beta < 1 - \frac{c_2}{\log qT},$$

for some positive constant c_2, by (4.12) and the range constraint of T. This gives us

$$\begin{aligned} |x^\rho| = x^\beta &< x^{1-c_2/\log(qT)}, \\ &= x \cdot \exp(\log x^{-(c_2/\log(qT))}), \\ &= x \exp(-c_2 \log x/\log(qT)). \end{aligned}$$

If we now consider the sum $\sum \frac{1}{|\rho|}$ we can split it into two parts: when $|\gamma| < 1$ and when $|\gamma| > 1$. We can estimate

$$\begin{aligned} \sum_{|\gamma|>1} \frac{1}{|\rho|} &= \int_1^T t^{-1} dN(t,\chi), \\ &= \left[\frac{N(t,\chi)}{t}\right]_1^T - \int_1^T \frac{N(t,\chi)}{t^2}\, dt, \\ &\ll \log(qT) + \int_1^T \frac{\log(qT)}{t}\, dt \quad \text{using } N(t,\chi) \ll t\log(qt), \\ &\ll \log^2(qT) \ll \log^2(qx). \end{aligned}$$

Also

$$\begin{aligned} \sum_{|\gamma|<1} \frac{1}{|\rho|} &\ll \sum_{|\gamma|<1} \log q, \\ &\ll \int_0^1 \log q\, dN(t,\chi), \\ &\ll [\log qN(t,\chi)]_0^1 \ll \log^2 q. \end{aligned}$$

The result of this is that

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + R_4(x,T), \tag{4.15}$$

where

$$\begin{aligned}|R_4(x,T) &\ll \left|\sum_{\rho}{}' \frac{x^\rho}{\rho} + R_3(x,T)\right|,\\ &\ll x\exp[-c_2\log x/\log(qT)]\cdot\sum_{\rho}{}' \frac{x^\rho}{\rho} + R_3(x,T),\\ &\ll x\log^2(qT)\exp[-c_2\log x/\log(qT)] + xT^{-1}\log^2(qx) + x^{\frac{1}{4}}\log x.\end{aligned}$$

Davenport then shows that if we suppose

$$q \leq \exp[C(\log x)^{\frac{1}{2}}], \tag{4.16}$$

where C is any positive constant and choose

$$T = \exp[C(\log x)^{\frac{1}{2}}], \tag{4.17}$$

then we have that

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\{x\exp[-C'(\log x)^{\frac{1}{2}}]\}, \tag{4.18}$$

for each non-principal χ to the modulus q.

This is the best bound known and is achieved by the careful selection of the constraints (4.16) and (4.17). We will see that this bound is sufficient in the proving of the conjecture, however it is interesting to explore how lightly one can constrain q and still achieve this goal. The following Proposition would seem to give the most freedom to q that we can permit it.

Proposition 4.2.2. *Suppose that*

$$q \leq \exp[C(\log x)^{1-\epsilon}] \quad \textit{and take} \quad T = \exp[C(\log x)^{\epsilon}],$$

where $0 < \epsilon\ (< 1/2)$ *and* $C \geq \log 2$. *Then for sufficiently large* x, *we have*

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\{x\exp[-C'(\log x)^{\epsilon}\}, \tag{4.19}$$

for some $C' > 0$.

Proof. We begin by recalling (4.15) and examining the components of the error term.

$$|R_4(x,T)| \ll x\log^2(qT)\exp[-c_2\log x/\log(qT)] + xT^{-1}\log^2(qx) + x^{\frac{1}{4}}\log x.$$

Under the constraints of the Proposition, the first term becomes

$$\begin{aligned} x\log^2(qT)\exp[-c_2\log x/\log(qT)] &\ll \frac{x(\log x)^{2(1-\epsilon)}}{e^{c_2\log x/\log(\exp[C(\log x)^{1-\epsilon}+C(\log x)^{\epsilon}])}}, \\ &\ll \frac{x(\log x)^{2(1-\epsilon)}}{e^{\frac{c_2}{2C}(\log x)^{\epsilon}}}, \\ &\ll x\exp[-C'(\log x)^{\epsilon}] \quad \text{for some } 0<C'<c_2/2C. \end{aligned}$$

Also

$$\begin{aligned} xT^{-1}\log^2(qT) &\ll x(\log e^{c(\log x)^{1-\epsilon}})^2\exp[-C(\log x)^{\epsilon}], \\ &\ll x(\log x)^{2(1-\epsilon)}\exp[-C(\log x)^{\epsilon}], \\ &\ll x\exp[-C''(\log x)^{\epsilon} \quad \text{for some } 0<C''<C. \end{aligned}$$

The last term is trivially bounded in the same way. □

The reader will note that the value of T here does not violate the constraint that $2 \le T \le x$ since

$$\exp[C(\log x)^{\epsilon}] \ll x,$$

for large x. Also, for $x \ge 3$

$$\exp[C(\log x)^{\epsilon}] \ge 2,$$

if and only if

$$\epsilon \ge \frac{\log\log 2 - \log C}{\log\log x},$$

which is true by the assumption that $C \ge \log 2$.

Let us consider how much freedom this really gives q. Examine the condition

$$q \le \exp[C(\log x)^{1-\epsilon}].$$

If we take $C = 1$, which we can since the only constraint on C is $C \ge \log 2$, then the comparison of

$$x \quad \text{with} \quad \exp[C(\log x)^{1-\epsilon}],$$

is equivalent to comparing

$$\log\log x \quad \text{with} \quad (1-\epsilon)\log\log x.$$

Clearly, as x becomes large, the difference between these two quantities can become arbitrarily large and thus we are excluding q from a growing number of values $\le x$. Therefore, the extra freedom that we have given to q is not as generous as it might seem.
The next section given an insight as to how this problem might be resolved.

4.2.1 The Search For q

Since Proposition 4.2.2 allows q only superficial freedom when considering large x we proceed, restricting q by

$$q \leq \exp[C(\log x)^{\frac{1}{2}}],$$

so that we may employ the strongest known error term for $\psi(x, \chi)$.

Now, assuming that $L(x, \chi)$ has no 'exceptional' zero, we have

$$\psi(x, \chi) \ll \frac{x}{e^{c(\log x)^{\frac{1}{2}}}}, \tag{4.20}$$

where c is some positive constant. Before continuing we give the following Proposition which will be useful in the subsequent calculations.

Proposition 4.2.3.

$$\frac{1}{e^{c(\log t)^{\frac{1}{2}}}} \ll \frac{1}{(\log t)^N},$$

for any $N > 0$.

Proof. Consider

$$(\log t)^N < e^{c(\log t)^{\frac{1}{2}}},$$

which is equivalent to

$$\left(\frac{N}{c}\right)^2 < \frac{\log t}{(\log\log t)^2}, \ (t > e).$$

This is clearly true for large enough t. □

Our crucial quantity (LHS below) can be manipulated as follows

$$\sum_{\substack{n \leq x \\ n \text{ prime}}} \chi(n) = \int_2^x \frac{1}{\log t}\, dS(t), \text{ where } S(t) = \psi(t, \chi). \tag{4.21}$$

We must now be careful as to how we bound $S(t)$. For $q < \exp[c(\log t)^{1/2}]$ or equivalently $t > \exp[(\frac{1}{c}\log q)^2]$, we have

$$S(t) \ll \frac{x}{e^{c(\log x)^{\frac{1}{2}}}} \ll \frac{x}{(\log x)^N},$$

by (4.20) and Proposition 4.2.3. If $t < \exp[(\frac{1}{c}\log q)^2] = x_q$ then we use the trivial bound

$$S(t) \ll t.$$

With this consideration, (4.21) becomes

$$\begin{aligned}
\sum_{\substack{n\leq x\\ n \text{ prime}}} \chi(n) &\ll \sum_{\substack{n\leq x_q\\ n \text{ prime}}} \chi(n) + \sum_{\substack{x_q\leq n\leq x\\ n \text{ prime}}} \chi(n),\\
&\ll \left[\frac{t}{\log t}\right]_2^{x_q} + Li_2\left(x_q\right) + \left[\frac{t}{e^{c(\log t)^{\frac{1}{2}}}\log t}\right]_{x_q}^{x} + \int_{x_q}^{x} \frac{1}{(\log t)^{N+2}}\, dt,\\
&\ll \frac{e^{(\frac{1}{c}\log q)^2}}{(\frac{1}{c}\log q)^2} + \frac{e^{(\frac{1}{c}\log q)^2}}{(\frac{1}{c}\log q)^4} + \frac{x}{(\log x)^{N+1}} + \frac{e^{(\frac{1}{c}\log q)^2}}{q(\frac{1}{c}\log q)^2} + \int_2^{x} \frac{1}{(\log t)^{N+2}}\, dt,\\
&\ll \frac{e^{(\frac{1}{c}\log q)^2}}{(\frac{1}{c}\log q)^2} + \frac{x}{(\log x)^{N+1}} + \frac{x}{(\log x)^{N+2}}, \qquad (4.22)
\end{aligned}$$

for $N > 0$ where we have used Proposition 4.2.3 and the relation

$$Li_N(x) = \frac{x}{(\log x)^N} + O\left(\frac{x}{(\log x)^{N+1}}\right). \qquad (4.23)$$

Now, applying the constraint

$$q \leq \exp[c'(\log x)^{1/2}],$$

where we choose $0 < c' < c$, the first term in (4.22) can be bounded as follows.

$$\begin{aligned}
\frac{e^{(\frac{1}{c}\log q)^2}}{(\frac{1}{c}\log q)^2} &\ll \frac{x^{(\frac{c'}{c})^2}}{(\frac{c'}{c}(\log x)^{\frac{1}{2}})^{2N}}, \text{ for } N > 0\\
&\ll \frac{x}{(\log x)^N}.
\end{aligned}$$

The result of this is that

$$\sum_{\substack{n\leq x\\ n \text{ prime}}} \chi(n) \ll \frac{x}{(\log x)^N}, \qquad (4.24)$$

for non-negative N. We can now apply (4.24) to the terms in (4.2).

The second term becomes

$$\begin{aligned}\sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} 1 \sum_{\substack{a\le x/b\\ a \text{ prime}}} \chi(a) &\ll \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \frac{(x/b)}{(\log(x/b))^{N+1}}, \quad \text{for any } N > -1,\\ &\ll \frac{x}{(\log x^{1/2})^{N+1}} \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \frac{1}{b},\\ &\ll \frac{x}{(\log x)^{N+1}} \left(\int_2^{\sqrt{x}} \frac{1}{t}\, dt + 1 \right),\\ &\ll \frac{x}{(\log x)^N}, \quad \text{where we take } N > 1.\end{aligned}$$

The third term in (4.2) is tackled as follows.

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \cdot \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} 1 \ll \frac{x^{1/2}}{(\log x)^N} \cdot \frac{x^{1/2}}{\log x} \ll \frac{x}{(\log x)^{N+1}}.$$

Finally, the first term can be written

$$\begin{aligned}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \sum_{\substack{b\le x/a\\ b \text{ prime}}} 1 &= \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) Li(x/a) + O\left(\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} |\chi(a)| \frac{(x/a)}{e^{D(\log(x/a))^{1/2}}} \right), \quad \text{where } D > 0 \text{ is a constant,}\\ &= \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) Li(x/a) + O\left(\frac{x}{(\log x)^{N+1}} \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{|\chi(a)|}{a} \right),\\ &= \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) Li(x/a) + O\left(\frac{x}{(\log x)^{N+1}} \int_2^{\sqrt{x}} \frac{1}{t}\, dt \right),\\ &= \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) Li(x/a) + O\left(\frac{x}{(\log x)^N} \right), \quad \text{where we take } N > 0.\end{aligned} \tag{4.25}$$

Here we have used the following result, proved by La Valle Poussin in 1899.

$$\pi(x) = Li(x) + O\left(e^{-a(\log x)^{1/2}} \right).$$

These three calculations similarly yield the same results for the sum $\sum_{\substack{ab\le x\\ a,b \text{ prime}}} \chi(b)$. We now deal with the remaining term in (4.1). Namely

$$\sum_{\substack{ab\le x\\ a,b \text{ prime}}} \chi(ab) = \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \sum_{\substack{b\le x/a\\ a \text{ prime}}} \chi(b) + \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \chi(b) \sum_{\substack{a\le x/b\\ a \text{ prime}}} \chi(a) - \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \cdot \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \chi(b).$$

Applying (4.24) to first term here yields

$$\begin{aligned}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \sum_{\substack{b\le x/a\\ a \text{ prime}}} \chi(b) &\ll \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \frac{(x/a)}{(\log(x/a))^{N+1}},\\ &\ll \frac{x}{(\log x)^{N+1}} \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{1}{a},\\ &\ll \frac{x}{(\log x)^N}, \quad \text{where we take } N>1. \end{aligned} \tag{4.26}$$

We note that (4.26) and (4.25) can both be improved so that the final error term is $O(x/\exp[A(\log x)^{1/2}])$ but this is not necessary since they will both be absorbed later on in a larger error term. Again, this can be applied to the second term with the same results simply by interchanging a and b. The last term becomes

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a) \cdot \sum_{\substack{b\le\sqrt{x}\\ b \text{ prime}}} \chi(b) \ll \left(\frac{x^{1/2}}{(\log x)^N}\right)^2 \ll \frac{x}{(\log x)^{2N}}.$$

Thus, for any $M>0$, we have

$$\frac{\#\{ab\le x;\ \chi(a)=\chi(b)=1\}}{\frac14\#\{ab\le x;\ (ab,d)=1\}} = 1+\frac{\log x}{x\log\log x}\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a)Li(x/a)+O\left(\frac{1}{\log\log x(\log x)^M}\right).$$

Now let us examine

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a)Li(x/a) = \sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{x\chi(a)}{a\log(x/a)} + O\left(x\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a(\log(x/a))^2}\right).$$

We have already shown that the last term is $o(x/\log x)$ by (4.6). We can change the summand of the first sum exactly as we did in (4.7) and only incurr a small error term that is $O(x/\log x)$.

The result of this is that

$$\sum_{\substack{a\le\sqrt{x}\\ a \text{ prime}}} \chi(a)Li(x/a) = \frac{x}{\log x}\sum_{a \text{ prime}} \frac{\chi(a)}{a} - \frac{x}{\log x}\sum_{\substack{a>\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a} + O\left(\frac{x}{\log x}\right).$$

Again, assuming $q \le \exp[C(\log x)^{\frac{1}{2}}]$ and that $L(s,\chi)$ has no exceptional zero then we have that

$$\sum_{\substack{a>\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a} = \int_{\sqrt{x}}^{\infty} \frac{1}{t}\, dS(t), \quad \text{where } S(t) = \sum_{\substack{p\le t\\ p \text{ prime}}} \chi(p). \tag{4.27}$$

Now

$$\begin{aligned}\sum_{\substack{p\le t\\ p \text{ prime}}} \chi(p) &\ll \psi(t,\chi),\\ &\ll \frac{t}{e^{c(\log t)^{\frac{1}{2}}}} \quad \text{for some } c>0,\\ &\ll \frac{t}{(\log t)^N} \quad \text{for any } N>0. \end{aligned}\tag{4.28}$$

On using (4.28), (4.27) becomes

$$\begin{aligned}\sum_{\substack{a>\sqrt{x}\\ a \text{ prime}}} \frac{\chi(a)}{a} &\ll \left[\frac{1}{(\log t)^N}\right]_{\sqrt{x}}^{\infty} + \int_{\sqrt{x}}^{\infty} \frac{1}{t(\log t)^N}\, dt,\\ &\ll \frac{1}{\log x} + \int_{\sqrt{x}}^{\infty} \frac{1}{t(\log t)^N}\, dt,\\ &\ll \frac{1}{\log x},\end{aligned}$$

where we have chosen $N=2$.

The calculations above allow us to assert that, for large x

$$\frac{\#\{ab\le x;\ \chi(a)=\chi(b)=1\}}{\frac{1}{4}\#\{ab\le x;\ (ab,d)=1\}} = 1 + \frac{\mathcal{L}_\chi + O(1)}{\log\log x}. \tag{4.29}$$

4.3 New Results

All that is then needed to prove the Conjecture is the following Theorem.

Theorem 4.3.1 (Modified Version Of Theorem 5b From [7]). *For large D, there are at least $D^{\frac{1}{10}}$ square-free integers $d \leq D$ such that $d \equiv 1 \mod 4$ and*

$$L(1, \chi_d) \geq e^{\gamma} \log\log D,$$

where $L(s, \chi_d)$ has no exceptional zero.

We recall that given a square free integer, congruent to 1 mod 4, there always exists a real, primitive character of this modulus..
The proof of the Theorem is a slight adaptation of that given in [7]. In the proof of Lemma 10.1 we there restrict our attention to primes $p \equiv 1 \mod 4$. Then Lemma 10.1 follows with both primes in the statement $p, q \equiv 1 \mod 4$. This in turn ensures that the d's in Lemmas 10.2 and 10.3 are all $\equiv 1 \mod 4$. Note, the fact that $L(s, \left(\frac{d}{\cdot}\right))$ has no exceptional zero follows from the proof of Lemma 10.3. Also we have $\left(\frac{d}{\cdot}\right) = \left(\frac{\cdot}{d}\right)$ as $d \equiv 1 \mod 4$.

It is noted in [3] that

$$\sum_p \frac{\chi(p)}{p} = \sum_{m \geq 1} \frac{\mu(m)}{m} \log L(m, \chi^m) = \log L(1, \chi) + E(\chi), \tag{4.30}$$

where

$$\sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) = -0.315718... \leq E(\chi) \leq \sum_p \left(\log\left(1 + \frac{1}{p}\right) - \frac{1}{p} \right) = -0.18198...$$

In light of (4.30) it is clear that the characters identified by Theorem 4.3.1 give rise to the extreme values of $\mathcal{L}_{\chi_d}$ we are looking for. Namely when

$$\mathcal{L}_{\chi_d} = \log\log\log x + O(1).$$

Since the results of the previous Section require that d be restricted by $d \leq \exp[C(\log x)^{1/2}]$, let us define

$$D(x) = \exp[C(\log x)^{1/2}].$$

Using this notation we have, on using (4.29) and Theorem 4.3.1, that the Conjecture is proved.

Theorem 4.3.2. *For large x, there exist $D(x)^{\frac{1}{10}}$ integers $d \leq D(x)$ for which*

$$\frac{\#\{ab \leq x;\ \chi_d(a) = \chi_d(b) = 1\}}{\frac{1}{4}\#\{ab \leq x;\ (ab, d) = 1\}} \quad \textit{is as large as} \quad 1 + \frac{\log\log\log x + O(1)}{\log\log x}.$$

Also, adapting the proof of Theorem 5b from [7] in a similar fashion to that used to produce Theorem 4.3.1, we have the following.

Theorem 4.3.3 (A Second Modified Version Of Theorem 5b From [7])**.** *For large D, there are at least $D^{\frac{1}{10}}$ square-free integers $d \leq D$ such that $d \equiv 1 \mod 4$ and*

$$L(1, \chi_d) \leq \frac{\pi^2}{6e^{\gamma} \log\log D},$$

where $L(s, \left(\frac{d}{\cdot}\right))$ has no exceptional zero.

This Theorem, along with (4.29) and (4.30) allows us to state the following Theorem.

Theorem 4.3.4. *For large x, there exist $D(x)^{\frac{1}{10}}$ integers $d \leq D(x)$ for which*

$$\frac{\#\{ab \leq x;\ \chi_d(a) = \chi_d(b) = 1\}}{\frac{1}{4}\#\{ab \leq x;\ (ab, d) = 1\}} \quad \textit{is as small as} \quad 1 - \frac{\log\log\log x + O(1)}{\log\log x}.$$

We now claim that the bias seen in Theorem 4.3.2 is indeed the largest such bias. In 1919, Littlewood proved the following Theorem.

Theorem 4.3.5. *Assume GRH. For any non-principal primitive character χ (mod q), one has*

$$|L(1, \chi)| \leq (2e^{\gamma} + o(1)) \log\log q.$$

Theorem 4.3.5 combines with Theorem 4.3.1 to support the following Conjecture.

Conjecture 4.3.1.

$$\max_{d \leq x} \left| \frac{\#\{ab \leq x;\ \chi_d(a) = \chi_d(b) = 1\}}{\frac{1}{4}\#\{ab \leq x;\ (ab, d) = 1\}} - 1 \right| \sim \frac{\log\log\log x}{\log\log x}.$$

We note that the above Conjecture holds under GRH. To see this, observe that by (4.8),

$$\left| \frac{\#\{ab \leq x;\ \chi_d(a) = \chi_d(b) = 1\}}{\frac{1}{4}\#\{ab \leq x;\ (ab, d) = 1\}} - 1 \right| = \frac{\mathcal{L}_{\chi} + O(1)}{\log\log x}, \quad \text{for } d \leq x.$$

Then Theorem 4.3.1 gives us that, for large x, there exists $d \leq x$ such that

$$e^{\gamma} \log\log x \leq L(1, \chi_d),$$

Also, Theorem 4.3.5 ensures that

$$L(1, \chi_d) \overset{\text{GRH}}{\leq} (2e^{\gamma} + o(1)) \log\log x, \quad \text{for any } d \leq x.$$

Therefore we have

$$\max_{d \leq x} \mathcal{L}_{\chi_d} \sim \log\log\log x + O(1).$$

Thus, assuming GRH, Conjecture 4.3.1 holds for the range $d \leq x$.

Appendix A

SageMath Code

Here I include my code that was used to perform various calculations in looking for extreme values of $\mathcal{L}_\chi$. By no means do I claim that this code is optimal or indeed efficiently implemented but it was more than adequate for the intended purpose.

The following code searches for real primitive characters with conductor $d \leq 500000$ that give extreme values of $\mathcal{L}_\chi$.

```
list_a=[]
interestBound = -1.5
m=prime_range(1,100000)
for n in range(1,500000):
        if ((n%4==1) & ((moebius(n)==0)==False)) or ((n%4==0)&
        ((moebius(n/4)==0)==False)&((n%4==2)or(n%4==3))):
            list_a=list_a+[n]
        if ((-n%4==1) & ((moebius(-n)==0)==False)) or ((-n%4==0)&
        ((moebius(-n/4)==0)==False)&((-n%4==2)or(-n%4==3))):
            list_a=list_a+[-n]
print ``list_a computed"

min = -1.5
for n in list_a:
    M = N(sum([kronecker(n,x)/x for x in m]))
    if (M < interestBound):
        print ``The Character given by X=(",n, "/.) gives L_X=", M
    if (M < min):
        min = M
        print ``min=", M
```

The following function takes a modulus d and a number x, and calculates

$$r_d(x) = \frac{\#\{pq \leq x : \chi_d(p) = \chi_d(q) = -1\}}{\frac{1}{4}\#\{pq \leq x : (pq, d) = 1\}}.$$

```
F=1, list_pqnums=[], list_pqsqfreefacs=[], list_pqsqfree=[]
list_s=[], list_den=[], p=1, q=1, r=1

def r(x,d):
    list_pqnums=[]
    list_pqsqfreefacs=[]
    list_pqsqfree=[]
    list_num=[]
    list_den=[]
    for n in range(1,x):
        F=factor(n)
        if (len(list(F))==2):
            list_pqnums=list_pqnums+[n]
    for m in list_pqnums:
        if ((moebius(m)==0)==False):
            list_pqsqfree=list_pqsqfree+[m]
            list_pqsqfreefacs=list_pqsqfreefacs+[list(factor(m))]
    print "Computed products pq up to", x, "where p and q are prime."
    for t in list_pqsqfreefacs:
        p=t[0][0]
        q=t[1][0]
        if ((kronecker(d,p)==-1)==True)&((kronecker(d,q)==-1)==True):
            list_num=list_num+[p*q]
    print "Computed products pq up to", x, "where p and q are prime
    and X(p)=X(q)=-1. There are", len(list_num), "if them."
    for s in list_pqsqfree:
        if gcd(s,d)==1:
            list_den=list_den+[s]
    print "Computed products pq up to", x, "where gcd(pq,d)=1. There
     are", len(list_den), "of them."
    r=N(len(list_num)/(0.25*len(list_den)))
    print "Therefore r_d(x)=", r
```

The code below gives the percentage for which the specified polynomial produces a prime over the first 10 million natural numbers.

```
c=0
for n in range (1,1000000):
    if is_prime(abs(n^2+n-1354363))==True:
        c=c+1
print N(100*(c/1000000))
```

Finally SageMath was also used to compute the number of primes of the forms $4n+1$ and $4n+3$ for various values of x up to four million. The code is straightforward.

```
def count(x):
    list_a=prime_range(2,x+1)
    n_1=0
    n_3=0
    for n in list_a:
        if (n%4)==1:
            n_1=n_1+1
        if (n%4)==3:
            n_3=n_3+1
    return [x,n_3,n_1]
list_b=[]
list_c=[100,200,300,400,500,600,700,800,900,1000,2000,3000,4000,5000,
6000,7000,8000,9000,10000,20000,50000,100000,200000,300000,400000,
500000,1000000,2000000,3000000,4000000]

for m in list_c:
    list_b=list_b+[count(m)]
print table(list_b)

latex(table(list_b))
```

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